TOPOLOGICAL CONJUGACY OF AFFINE TRANSFORMATIONS OF COMPACT ABELIAN GROUPS

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0. **Introduction.** We consider the following problem. If X and Y are compact connected metric abelian groups, $T=a+A$ an affine transformation of X and $S=b+B$ an affine transformation of Y, what are necessary and sufficient conditions for every continuous mapping g of X onto Y satisfying $gT=Sg$ to be affine? Sufficient conditions are obtained in Theorem 3 in the case when the character group $\hat{Y}$ of Y is polynomially annihilated by B (see Definition 1). In Theorem 6 we show that these conditions are also necessary when Y is a finite-dimensional group and in Theorem 8 we state (without proof) that this is also true in a more general setting. An example is given to show the assumption that $\hat{Y}$ be polynomially annihilated by B cannot be dropped from Theorem 3. We also give an example which shows Theorem 6 to be false if Y is not finite-dimensional but $\hat{Y}$ is polynomially annihilated by B.

Theorem 7 deals with the case when T and S are endomorphisms of an n-dimensional group. A conjugacy property of affine transformations with quasi-discrete spectrum is given in Theorem 4, and Theorem 5 gives information on continuous roots of affine transformations.

The results of this paper were proved in [10] for the cases when X and Y are finite-dimensional tori. The idea of using Theorem 1 was obtained from the paper [2].

1. **Definitions and notations.** Let Y be a compact connected metric abelian (c.c.m.a.) group. We shall use additive notation in such groups. $\hat{Y}$ will denote the discrete torsion-free countable abelian character group of Y, and multiplicative notation will be used in $\hat{Y}$. Y can be written as an inverse limit inv lim (Y_m, σ_m), where each Y_m $(m \geqq 1)$ is a finite-dimensional torus and σ_m is a homomorphism of Y_{m+1} onto Y_m. If Y is n-dimensional then each Y_m can be chosen to be an n-dimensional torus.

An affine transformation S of a c.c.m.a. group Y is a transformation of the form $S(y)=b+B(y)$, $y \in Y$, where $b \in Y$ and B is an endomorphism of Y onto Y. We write $S=b+B$. Every affine transformation of Y is continuous and preserves Haar measure. An endomorphism B of Y onto Y induces a one-to-one dual endomorphism, which we also denote by B, of $\hat{Y}$ into $\hat{Y}$ defined by $(B\gamma)(y)=\gamma(By)$, $y \in Y$, $\gamma \in \hat{Y}$.

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The affine transformation $S=b+B$ of Y is ergodic if and only if $B^n\gamma=\gamma$, $\gamma \in \hat{Y}$, $n>0$, implies $B\gamma=\gamma$, and $[b, (B-I)Y]=Y$ where $[b, (B-I)Y]$ denotes the smallest closed subgroup of Y containing b and $(B-I)Y$. (I is the identity mapping of Y [6].) Also, $S=b+B$ is ergodic if and only if there exists $y_0 \in Y$ such that $\{S^n(y_0) \mid n \geqq 0\}$ is dense in Y [10]. From the first condition it follows that an endomorphism B of Y onto Y is ergodic if and only if $B^n\gamma=\gamma$, $\gamma \in \hat{Y}$, $n>0$, implies $\gamma=1$ (see also [5]). Also from the first condition we have that $S=b+B$ is strong mixing if and only if B is ergodic (see also [3]).

$\boldsymbol{R}^n$ will denote real Euclidean n-space, $\boldsymbol{Z}^n$ the subgroup of $\boldsymbol{R}^n$ of points with integer coordinates and $\boldsymbol{K}^n=\boldsymbol{R}^n/\boldsymbol{Z}^n$ the n-dimensional torus. $\mathscr{R}(\mu)$ and $\mathscr{I}(\mu)$ will denote the real and imaginary parts of the complex number μ, and if $P(x)=(P_1(x), P_2(x), \ldots, P_n(x))$ is a transformation from a set X to complex n-space $\boldsymbol{C}^n$, then $\mathscr{R}P$ and $\mathscr{I}P$ will denote the transformations of X to $\boldsymbol{R}^n$ defined by $(\mathscr{R}P)(x)=(\mathscr{R}P_1(x), \ldots, \mathscr{R}P_n(x))$ and $(\mathscr{I}P)(x)=(\mathscr{I}P_1(x), \ldots, \mathscr{I}P_n(x))$ respectively.

$\boldsymbol{Q}$ will denote the field of rational numbers and $\boldsymbol{Q}[\theta]$ the algebra of all polynomials in θ with coefficients from $\boldsymbol{Q}$.

2. Preliminary results.

THEOREM 1 (VAN KAMPEN). *Let Y be a c.c.m.a. group and f be a continuous function from Y to the set of complex numbers of unit modulus. Then f can be expressed in the form $f(y)=\alpha(y)e^{i\phi(y)}$, $y \in Y$, where $\alpha \in \hat{Y}$ is uniquely determined by f, $\phi\colon Y \to R$ is continuous and is uniquely determined up to an additive constant.*

Proofs of this theorem can be found in [9] and [2]. The following is immediate from Theorem 1.

COROLLARY 1.1. *Let X and Y be c.c.m.a. groups and $g\colon X \to Y$ a continuous mapping. For each $\gamma \in \hat{Y}$ there exists a unique $\alpha_\gamma \in \hat{X}$ and a continuous mapping $\phi_\gamma\colon X \to R$ unique up to an additive constant, such that* $(\gamma \circ g)(x)=\alpha_\gamma(x)\exp[i\phi_\gamma(x)]$, $x \in X$. *Furthermore* $\exp[i\phi_{\gamma\gamma^1}(x)]=\exp[i\phi_\gamma(x)]\cdot\exp[i\phi_{\gamma^1}(x)]$, $x \in X$, $\gamma, \gamma^1 \in \hat{Y}$.

THEOREM 2. *Let X and Y be c.c.m.a. groups and suppose that for every $\gamma \in \hat{Y}$ there exists a continuous mapping $\phi_\gamma\colon X \to R$ such that $\phi_{\gamma\gamma^1}=\phi_\gamma+\phi_{\gamma^1}$, $\gamma, \gamma^1 \in \hat{Y}$. Then there exists a continuous mapping $u\colon X \to Y$ such that $\gamma \circ u(x)=\exp[i\phi_\gamma(x)]$, $x \in X$, $\gamma \in \hat{Y}$, and u is homotopic to a constant.*

Proof. For each $x \in X$ the mapping $\gamma \to \exp[i\phi_\gamma(x)]$ is a character of $\hat{Y}$ and therefore there exists $y_x \in Y$ such that $\gamma(y_x)=\exp[i\phi_\gamma(x)]$. Define $u\colon X \to Y$ by $u(x)=y_x$. u is clearly continuous.

For each $x \in X$ and each $t \in [0, 1]$ the mapping $\gamma \to \exp[it\phi_\gamma(x)]$ is a character of $\hat{Y}$ and, as above in the case $t=1$, there exists a continuous mapping $u_t\colon X \to Y$ such that $\gamma(u_t(x))=\exp[it\phi_\gamma(x)]$, $x \in X$, $\gamma \in \hat{Y}$. u_t is a homotopy between u and a constant.

3. **Topological conjugacy and groups with polynomially annihilated character groups.** Let Y be a c.c.m.a. group and B an endomorphism of Y onto Y. Let $p(\theta)=n_0+n_1\theta+\cdots+n_k\theta^k$ be a polynomial over $\boldsymbol{Z}$. We shall say that p is an annihilating polynomial of $\gamma\in\hat{Y}$ with respect to B if $\gamma^{n_0}\cdot B\gamma^{n_1}\cdots B^k\gamma^{n_k}=1$.

Suppose $\gamma\in\hat{Y}$ has a nontrivial annihilating polynomial with respect to B. Let M_γ denote the set of all polynomials over $\boldsymbol{Q}$ some integral multiple of which is an annihilating polynomial of γ with respect to B. M_γ is an ideal in $\boldsymbol{Q}[\theta]$ and therefore there exists a unique monic polynomial $q_\gamma\in\boldsymbol{Q}[\theta]$ such that M_γ is the principal ideal generated by q_γ [8, p. 121]. If $q_\gamma(\theta)=s_0+s_1\theta+\cdots+s_{l-1}\theta^{l-1}+\theta^l$ then $s_0\neq 0$ for otherwise $q_\gamma^1(\theta)=s_1+s_2\theta+\cdots+s_{l-1}\theta^{l-2}+\theta^{l-1}$ would be a monic polynomial generating M_γ. If n_q is the lowest common denominator of the nonzero members of $s_0, s_1,\ldots, s_{l-1}$ then $p_\gamma(\theta)=n_qs_0+n_qs_1\theta+\cdots+n_q\theta^l$ is a polynomial over $\boldsymbol{Z}$ which will be called the minimal annihilating polynomial of γ with respect to B.

DEFINITION 1. Let Y be a c.c.m.a. group and B an endomorphism of Y onto Y. We say that $\hat{Y}$ is polynomially annihilated by B if every element of $\hat{Y}$ has a nontrivial annihilating polynomial with respect to B.

If Y is an n-dimensional c.c.m.a. group then $\hat{Y}$ is polynomially annihilated by any endomorphism B of Y onto Y. This follows because $\hat{Y}$ is isomorphic to a subgroup of the additive group $\boldsymbol{Q}^n$ (the direct sum of n copies of $\boldsymbol{Q}$) and therefore the one-to-one endomorphism B of $\hat{Y}$ corresponds to an $n\times n$ matrix with rational entries and nonzero determinant. The Cayley-Hamilton theorem shows that some integral multiple of the characteristic polynomial of this matrix is an annihilating polynomial, with respect to B, of every element of $\hat{Y}$. If $q(\theta)=s_0+s_1\theta+\cdots+\theta^n$ is the characteristic polynomial of some matrix representation of B and if n_q is the lowest common denominator of the nonzero members of $s_0, s_1,\ldots, s_{n-1}$, then the polynomial $p(\theta)=n_qs_0+n_qs_1\theta+\cdots+n_q\theta^n$ is a polynomial over $\boldsymbol{Z}$, which will be called the annihilating polynomial of $\hat{Y}$ with respect to B. This polynomial is independent of the matrix representation of B.

The following lemma will be used in the proof of Theorem 3.

LEMMA 1. *Let X be a c.c.m.a. group and $T=a+A$ an affine transformation of X. Suppose $\Phi\colon X\to R^n$ is a nonconstant continuous function and M is a linear transformation of $\boldsymbol{R}^n$ such that $\Phi(Tx)=M\Phi(x)+d$, $x\in X$, where $d\in\boldsymbol{R}^n$. Then there exists $\delta\in\hat{X}$, $\delta\neq 1$, and a root λ, with $|\lambda|=1$, of the characteristic equation of M such that $A^p\delta=\delta$ for some $p\geqq 1$ and $\delta(a+A(a)+\cdots+A^{p-1}(a))=\lambda^p$ for all such p.*

Proof. We consider $\boldsymbol{R}^n$ as a subset of $\boldsymbol{C}^n$ (complex n-space) in the usual way and complexify M. There exists an invertible linear transformation U of $\boldsymbol{C}^n$ such that $U^{-1}MU=D_M$, the Jordan normal form of the linear transformation M. Therefore $U^{-1}\Phi(Tx)=D_MU^{-1}\Phi(x)+U^{-1}d$, $x\in X$ If $w_1, w_2,\ldots, w_n$ denotes the fixed basis of $\boldsymbol{R}^n$ then $w_1, w_2,\ldots, w_n$ is also a basis, using complex coefficients for $\boldsymbol{C}^n$. Suppose $U^{-1}\Phi(x)=\sum_{i=1}^n f_i(x)w_i$. Each $f_i\colon X\to\boldsymbol{C}$ is continuous, and if i_0 is the least positive integer for which f_{i_0} is nonconstant then $f_{i_0}(Tx)=\lambda f_{i_0}(x)+e$, $x\in X$, where $e\in\boldsymbol{C}$

and λ is an eigenvalue of M. If $l\colon X\to \boldsymbol{C}$ is defined by $l(x)=f_{i_0}(x)-\int_X f_{i_0}(x)\,dm$, where m denotes Haar measure on X, then $l(Tx)=\lambda l(x)$ and l is nonconstant and continuous. Since T maps X onto X, $\sup_X |l(Tx)|=|\lambda|\sup_X |l(x)|$ implies $|\lambda|=1$. But $l\in L^2(X)$ and therefore $l(x)=\sum_i b_i\delta_i(x)$ (L^2 convergence) where $\delta_i\in\hat{X}$ and $\sum_i |b_i|^2<\infty$. From the equation $l(T^px)=\lambda^p l(x)$, $p\geqq 1$, we have

$$\sum_i b_i\delta_i(a+Aa+\cdots+A^{p-1}a)\delta_i(A^px) = \lambda^p\sum_i b_i\delta_i(x)\quad (L^2 \text{ convergence}).$$

If δ_i, $A\delta_i$, $A^2\delta_i,\ldots$ are all distinct then $b_i=0$ for otherwise the condition $\sum_i |b_i|^2<\infty$ is violated. Therefore $b_i\neq 0$ implies $A^p\delta_i=\delta_i$ for some $p\geqq 1$ and when this occurs $\delta_i(a+Aa+\cdots+A^{p-1}a)=\lambda^p$. Since $l(x)$ is nonconstant there must be some $\delta_i\in\hat{X}$, $\delta_i\neq 1$, with this property.

THEOREM 3. *Let X and Y be c.c.m.a. groups. Let $T=a+A$ be an affine transformation of X and $S=b+B$ an affine transformation of Y. Suppose further that $\hat{Y}$ is polynomially annihilated by B. If there exists a nonaffine continuous mapping $g\colon X\to Y$ such that $gT=Sg$ then there exists $\delta\in\hat{X}$, $\delta\neq 1$, and a root λ, with $|\lambda|=1$, of the minimal annihilating polynomial with respect to B of some element of $\hat{Y}$, such that $A^p\delta=\delta$ for some $p\geqq 1$ and $\delta(a+Aa+\cdots+A^{p-1}a)=\lambda^p$ for all such p.*

Proof. Using the notation of Corollary 1.1, for $\gamma\in\hat{Y}$ let

$$(\gamma\circ g)(x) = \alpha_\gamma(x)\exp[i\phi_\gamma(x)],$$

where $\alpha_\gamma\in\hat{X}$ and $\phi_\gamma\colon X\to R$ is continuous. Since g is nonaffine there exists $\gamma_0\in\hat{Y}$ such that ϕ_{γ_0} is nonconstant. Applying $\gamma\in\hat{Y}$ to the equation $gT=Sg$ and using the uniqueness asserted in Corollary 1.1 we have $\alpha_\gamma(a)\exp[i\phi_\gamma(Tx)]=\gamma(b)\exp[i\phi_{B\gamma}(x)]$. Since X is connected this implies

$$\phi_\gamma(Tx) = \phi_{B\gamma}(x)+c_\gamma,\qquad x\in X,$$

where $c_\gamma\in\boldsymbol{R}$. Suppose that p_{γ_0}, the minimal annihilating polynomial of γ_0 with respect to B, is of degree n. Define $\Phi\colon X\to\boldsymbol{R}^n$ by

$$\Phi(x) = \begin{bmatrix}\phi_{\gamma_0}(x)\\ \phi_{B\gamma_0}(x)\\ \vdots\\ \phi_{B^{n-1}\gamma_0}(x)\end{bmatrix},\qquad x\in X.$$

Φ is nonconstant and continuous. If $p_{\gamma_0}(\theta)=m_0+m_1\theta+\cdots+m_n\theta^n$, $m_i\in\boldsymbol{Z}$ $(1\leqq i\leqq n)$, $m_n\neq 0$, then using the connectedness of X we have that $m_n\phi_{B^n\gamma_0}(x)+m_{n-1}\phi_{B^{n-1}\gamma_0}+\cdots+m_0\phi_{\gamma_0}(x)$ is a constant mapping. Let M denote the linear transformation of $\boldsymbol{R}^n$ given by the matrix

$$\begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -m_0/m_n & -m_1/m_n & \cdots & -m_{n-2}/m_n & -m_{n-1}/m_n \end{bmatrix}.$$

Then $\Phi(Tx)=M\Phi(x)+d$, $x\in X$, where $d\in \boldsymbol{R}^n$, and the result follows from Lemma 1 since p_{γ_0} is the characteristic polynomial of M.

COROLLARY 3.1. *Let X, Y, T, S be as in Theorem* 3 *with the additional assumption that T is ergodic. If there is a nonaffine continuous mapping $g:X\to Y$ such that $gT=Sg$ then there exists $\delta\in\hat{X}$, $\delta\neq 1$, and a root λ, with $|\lambda|=1$, of the minimal annihilating polynomial with respect to B of some element of $\hat{Y}$, such that λ is not a root of unity, $A\delta=\delta$ and $\delta(a)=\lambda$.*

Hence if T is strong mixing, all continuous mappings g: $X\to Y$ such that $gT=Sg$ are affine.

Proof. Let δ be the element of $\hat{X}$ and λ the complex number which are determined by Theorem 3. Since $A^p\delta=\delta$ for some $p\geqq 1$, the ergodicity of T implies $A\delta=\delta$ and hence $\delta(a)=\lambda$. If λ were a root of unity then since $[a,(A-I)X]=X$, δ would only assume a finite number of values on X and would have to be the identity character.

Lastly, if T is strong mixing then A is ergodic and there is no $\delta\in\hat{X}$, $\delta\neq 1$, with $A\delta=\delta$.

Theorem 2 of Arov [2] follows from Corollary 3.1.

The notion of a measure-preserving transformation with quasi-discrete spectrum has been defined by Abramov [1], and the notion of a homeomorphism with quasi-discrete spectrum has been defined by Hahn and Parry [4]. An ergodic affine transformation $S=b+B$ of a c.c.m.a. group Y has quasi-discrete spectrum as a (Haar) measure-preserving transformation if and only if it has quasi-discrete spectrum as a homeomorphism. In fact $S=b+B$, assumed to be ergodic, has quasi-discrete spectrum in either sense if and only if $\bigcap_{n=0}^{\infty}(B-I)^n Y=\{0\}$, where I denotes the identity mapping of Y [7]. The following result extends Theorem 6 of the paper [4].

THEOREM 4. *Let X and Y be c.c.m.a. groups and let $T=a+A$ be an ergodic affine transformation of X and $S=b+B$ an ergodic affine transformation of Y. If S has quasi-discrete spectrum then all continuous mappings g: $X\to Y$ satisfying $gT=Sg$ are affine.*

Proof. Let $\gamma\in\hat{Y}$. There exists $n\geqq 1$ such that $(\theta-1)^n$ is an annihilating polynomial of γ with respect to B. It follows that the roots of the minimal annihilating polynomial of γ with respect to B are equal to 1. The result follows from Corollary 3.1.

THEOREM 5. *Let Y be a c.c.m.a. group and $S=b+B$ a strong mixing affine transformation of Y such that $\hat{Y}$ is polynomially annihilated by B. Then every continuous pth root ($p \geqq 1$) of S is an affine transformation and S has a continuous pth root if and only if there is an endomorphism C of Y onto Y with $C^p=B$.*

Proof. Suppose g is a continuous pth root of S. Then $gS=Sg$ and g is affine by Corollary 3.1. Since S is strong mixing B is ergodic and therefore $(B-I)Y=Y$. Choose $y_0 \in Y$ so that $(B-I)y_0=b$ and the homeomorphism $h\colon Y \to Y$, defined by $h(y)=y_0+y$, satisfies $hS=Bh$. Therefore S has a continuous pth root if and only if B has a continuous pth root. Any continuous pth root of B is affine and the pth power of its endomorphism part will be B. Conversely if C is an endomorphism of Y onto Y with $C^p=B$ then C is a continuous pth root of B.

As a special case of Corollary 3.1 we have the following result. If Y is a c.c.m.a. group and B is an ergodic endomorphism of Y onto Y which polynomially annihilates $\hat{Y}$, then every continuous mapping commuting with B is affine. The example below shows that this result is false (and therefore Theorem 3 is false) if the assumption that $\hat{Y}$ be polynomially annihilated by B is dropped.

Let $\boldsymbol{K}^\infty$ denote the two-sided infinite-dimensional torus (i.e. the two-sided infinite direct sum of copies of $\boldsymbol{K}$) and let B denote the shift automorphism of $\boldsymbol{K}^\infty$ defined by $(Bz)_n=z_{n+1}$ if $z=(z_n)$. No nontrivial element of $\hat{\boldsymbol{K}}^\infty$ is polynomially annihilated by B. Let $f\colon \boldsymbol{K} \to \boldsymbol{K}$ be any homeomorphism and define $F\colon \boldsymbol{K}^\infty \to \boldsymbol{K}^\infty$ by $(F(z))_n=f(z_n)$, $-\infty<n<\infty$. F is a homeomorphism and $FB=BF$. Moreover F can be chosen to be nonaffine by choosing f nonaffine.

It would be interesting to know if the condition that $\hat{Y}$ be polynomially annihilated by B follows from the fact that every continuous mapping commuting with B (B ergodic) is affine.

4. Converses of Theorem 3.

LEMMA 2. *Let X and Y be c.c.m.a. groups and let them be represented as $X=\operatorname{inv}\lim(X_q, \tau_q)$ and $Y=\operatorname{inv}\lim(Y_m, \sigma_m)$ where X_q ($q \geqq 1$) and Y_m ($m \geqq 1$) are finite-dimensional tori. Let C be a homomorphism of X onto Y and let $u\colon X \to Y$ be a continuous mapping which depends only on X_{k_0} and which is homotopic to a constant by a homotopy which depends only on X_{k_0}. Then $C+u$ maps X onto Y.*

Proof. Let C_m and u_m ($m \geqq 1$) denote the mappings of X to Y_m obtained by projecting C and u onto Y_m. $C+u$ will map X onto Y if and only if C_m+u_m maps X onto Y_m for each $m \geqq 1$. For each $m \geqq 1$ there exists $q_m \geqq 1$ such that C_m only depends on X_{q_m}. Let $k_m=\max(q_m, k_0)$. Then C_m can be considered as a homomorphism of X_{k_m} onto Y_m and u_m can be considered as a continuous mapping of X_{k_m} into Y_m which is homotopic (on X_{k_m}) to a constant. The result will follow if we can show that whenever C is a homomorphism of $\boldsymbol{K}^n$ onto $\boldsymbol{K}^m$ and $u\colon \boldsymbol{K}^n \to \boldsymbol{K}^m$ is a continuous mapping homotopic to a constant then $C+u$ maps $\boldsymbol{K}^n$ onto $\boldsymbol{K}^m$. However this result follows from Lemma 1 of [10].

LEMMA 3. *Let $P: \boldsymbol{R}^n \to \boldsymbol{R}^n$ be a continuous mapping such that $P(v+\tau)=P(v)$, $v \in \boldsymbol{R}^n$, $\tau \in \boldsymbol{Z}^n$ and $\|P(v)-P(v')\| < \|v-v'\|$, $v, v' \in \boldsymbol{R}^n$ where $\|\cdot\|$ denotes the usual norm in $\boldsymbol{R}^n$. Let $\psi: \boldsymbol{K}^n \to \boldsymbol{K}^n$ be the continuous mapping defined by $\psi\pi=\pi P$, where $\pi: \boldsymbol{R}^n \to \boldsymbol{K}^n$ is the natural projection. Then $I+\psi$ is a one-to-one mapping of $\boldsymbol{K}^n$. (I denotes the identity mapping of $\boldsymbol{K}^n$.)*

Proof. Let I' denote the identity mapping of $\boldsymbol{R}^n$. $I'+P$ is a one-to-one mapping because $v+P(v)=v'+P(v')$ implies $v-v'=P(v')-P(v)$ and hence $v=v'$. Suppose $(I+\psi)\pi(v)=(I+\psi)\pi(v')$. Then $\pi(I'+P)(v)=\pi(I'+P)(v')$ and

$$\begin{aligned}(I'+P)(v) &= (I'+P)(v')+\tau, \qquad \tau \in \boldsymbol{Z}^n\\ &= (I'+P)(v'+\tau).\end{aligned}$$

Therefore $v=v'+\tau$ and $\pi(v)=\pi(v')$. This proves that $I+\psi$ is one-to-one.

The following theorem gives a converse to Theorem 3 in the cases when Y is a finite-dimensional group.

THEOREM 6. *Let X and Y be c.c.m.a. groups and suppose that Y is n-dimensional. Let $T=a+A$ be an affine transformation of X, $S=b+B$ an affine transformation of Y and suppose there exists a continuous mapping h of X onto Y such that $hT=Sh$. Suppose further there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a root λ, with $|\lambda|=1$, of the annihilating polynomial of $\hat{Y}$ with respect to B such that $A^p\delta=\delta$ for some $p \geqq 1$ and $\delta(a+A(a)+\cdots+A^{p-1}(a))=\lambda^p$ for all such p. Then there exists a nonaffine continuous mapping g of X onto Y such that $gT=Sg$. Moreover, if h is given to be a homeomorphism then g can be chosen to be a homeomorphism.*

Proof. We may as well assume that the given mapping h is affine or there is nothing to prove. Suppose $h=c+C$, where $c \in Y$ and C is a homomorphism of X onto Y. We shall use Theorem 2 and to do this we have to construct continuous mappings $\phi_\gamma: X \to R$ for each $\gamma \in \hat{Y}$.

Since Y is n-dimensional $\hat{Y}$ is isomorphic to a subgroup $\boldsymbol{Q}^n_Y$ of the additive group $\boldsymbol{Q}^n$ and we can choose $\boldsymbol{Q}^n_Y$ so that if $d_i=(d_{i1}, d_{i2}, \ldots, d_{in})$ where

$$\begin{aligned} d_{ij} &= 1 \quad \text{if } i = j,\\ &= 0 \quad \text{if } i \neq j,\end{aligned}$$

then $d_i \in \boldsymbol{Q}^n_Y$ $(1 \leqq i \leqq n)$. Let $[M]$ be the matrix (with rational entries) representing the action of B on $\boldsymbol{Q}^n_Y$, and let $\gamma_i \in \hat{Y}$ correspond under the above isomorphism to $d_i \in \boldsymbol{Q}^n_Y$ $(1 \leqq i \leqq n)$. Let M denote the linear transformation of $\boldsymbol{R}^n$ induced by the matrix $[M]$.

Suppose that p is the smallest positive integer such that $A^p\delta=\delta$. Define $f: X \to \boldsymbol{C}$ by

$$f(x) = \sum_{j=0}^{p-1} \frac{\delta(a+A(a)+\cdots+A^{j-1}(a))}{\lambda^j}\, \delta(A^j x), \qquad x \in X,$$

f is a nonconstant continuous function satisfying $f(Tx)=\lambda f(x)$, $x \in X$. If $w_1, w_2, \ldots, w_n$ denotes the fixed basis of $\boldsymbol{R}^n$ it is also a basis for $\boldsymbol{C}^n$. Let U be the invertible linear transformation of $\boldsymbol{C}^n$ such that $U^{-1}MU=D_M$, the Jordan normal form of the complexified linear transformation M. Let j_0 be the largest integer for which w_{j_0} corresponds to the eigenvalue λ of D_M. Then $U(f(x)w_{j_0})$ is nonconstant and so either $\mathscr{R}U(f(x)w_{j_0})$ or $\mathscr{I}U(f(x)w_{j_0})$ is nonconstant. Suppose, without loss of generality, that $\mathscr{R}U(f(x)w_{j_0})$ is nonconstant and define the mappings $\phi_{\gamma_i}: X \to \boldsymbol{R}$ by

$$\sum_{i=1}^{n} \phi_{\gamma_i}(x)w_i = \mathscr{R}U(f(x)w_{j_0}), \qquad x \in X.$$

Let $\gamma \in \hat{Y}$. If $\gamma^{m_0}=\gamma_1^{m_1}\cdot \gamma_2^{m_2}\cdot \ldots \cdot \gamma_n^{m_n}$, $m_0, m_1, \ldots, m_n \in \boldsymbol{Z}$, $m_0 \neq 0$, define $\phi_\gamma: X \to \boldsymbol{R}$ by

$$\phi_\gamma(x) = \frac{m_1}{m_0}\phi_{\gamma_1}(x)+\frac{m_2}{m_0}\phi_{\gamma_2}(x)+\cdots+\frac{m_n}{m_0}\phi_{\gamma_n}(x), \qquad x \in X.$$

Then $\phi_{\gamma\gamma^1}=\phi_\gamma+\phi_{\gamma^1}$, $\gamma, \gamma^1 \in \hat{Y}$. Also

$$\begin{aligned}\sum_{i=1}^{n} \phi_{\gamma_i}(Tx)w_i &= \mathscr{R}U(f(Tx)w_{j_0}) = \mathscr{R}UD_M(f(x)w_{j_0}) \\ &= M\sum_{i=1}^{n} \phi_{\gamma_i}(x)w_i = \sum_{i=1}^{n} \phi_{B\gamma_i}(x)w_i.\end{aligned}$$

Therefore $\phi_{B\gamma_i}(Tx)=\phi_{B\gamma_i}(x)$, $x \in X$, $1 \leqq i \leqq n$, and hence $\phi_\gamma(Tx)=\phi_{B\gamma}(x)$, $x \in X$, $\gamma \in \hat{Y}$. By Theorem 2 there exists a continuous mapping $u: X \to Y$ such that $\gamma(u(x)) =\exp[i\phi_\gamma(x)]$, $x \in X$, $\gamma \in \hat{Y}$, $u(Tx)=Bu(x)$, $x \in X$, and u is homotopic to a constant. Let $g: X \to Y$ be defined by $g(x)=c+C(x)+u(x)$, $x \in X$. $g(Tx)=c+C(Tx)+u(Tx) =S(c+C(x))+Bu(x)=Sg(x)$, $x \in X$.

It remains to show that g maps X onto Y. Suppose $X=\operatorname{inv}\lim(X_q, \tau_q)$ and $Y=\operatorname{inv}\lim(Y_m, \sigma_m)$ where the X_q $(q \geqq 1)$ are finite-dimensional tori and the Y_m $(m \geqq 1)$ are n-dimensional tori. Suppose the given character $\delta \in \hat{X}_{k_0}$. Then each mapping $\phi_\gamma: X \to \boldsymbol{R}$ only depends on X_{k_0} and therefore u only depends on X_{k_0} and is homotopic to a constant by a homotopy depending only on X_{k_0} (Theorem 2). The fact that g maps X onto Y now follows from Lemma 2.

We now show that if h is given to be a homeomorphism then g can be chosen to be a homeomorphism. Since we are assuming $h=c+C$, C will be an isomorphism of X onto Y. Let $g_t: X \to Y$, $t \in [0, 1]$, be defined by $g_t(x)=c+C(x)+u_t(x)$, $x \in X$, where $u_t: X \to Y$ satisfies $\gamma(u_t(x))=\exp[it\phi_\gamma(x)]$, $x \in X$, $\gamma \in \hat{Y}$ (Theorem 2). By Lemma 2 g_t is a continuous mapping of X onto Y and $g_tT=Sg_t$, $t \in [0, 1]$. We shall show that g_t is one-to-one for sufficiently small t.

It suffices to show that $g_t \circ C^{-1}: Y \to Y$ is one-to-one for sufficiently small t. We have $g_t \circ C^{-1}(y)=c+y+u_t \circ C^{-1}y$, $y \in Y$. Let k be the smallest integer for which $\delta \circ C^{-1} \in \hat{Y}_k$, where δ is the given element of $\hat{X}$. By the definition of ϕ_γ,

$\gamma \in \hat{Y}$, each $\phi_\gamma \circ C^{-1}$ can be considered as a real-valued function of Y_k, and therefore induces a mapping P_γ: $\boldsymbol{R}^n \to \boldsymbol{R}$ defined by $P_\gamma(v) = \phi_\gamma \circ C^{-1}(y_v)$, $v \in \boldsymbol{R}^n$, where y_v is any point of Y which has component $v + \boldsymbol{Z}^n$ in Y_k. Since each P_γ is a linear combination of sines and cosines of the coordinates of $\boldsymbol{R}^n$, there exists a constant N such that if β is a generator of any $\hat{Y}_m$ $(m \geqq 1)$ then

$$|P_\beta(v) - P_\beta(v')| \leqq N\|v - v'\|, \qquad v, v' \in \boldsymbol{R}^n,$$

where $\|\cdot\|$ denotes the usual norm in $\boldsymbol{R}^n$.

Choose $t_0 \in [0, 1]$ so that $nt_0N < 1$. Let $y, y' \in Y$, $y \neq y'$. Suppose $y = (y_1, y_2, \dots)$, $y' = (y'_1, y'_2, \dots)$ where $y_i, y'_i \in Y_i$, $i \geqq 1$. We shall show that $g_{t_0} \circ C^{-1}(y) \neq g_{t_0} \circ C^{-1}(y')$. If $y_k = y'_k$ then $\phi_\gamma \circ C^{-1}(y) = \phi_\gamma \circ C^{-1}(y')$, $\gamma \in \hat{Y}$, and therefore $u_{t_0} \circ C^{-1}(y) = u_{t_0} \circ C^{-1}(y')$. Hence $g_{t_0} \circ C^{-1}(y) - g_{t_0} \circ C^{-1}(y') = y - y' \neq 0$. Now suppose $y_k \neq y'_k$. Considering Y_k as an n-torus let $\beta_1, \beta_2, \dots, \beta_n \in \hat{Y}_k$ be defined by $\beta_j(z_1, \dots, z_n) = \exp(2\pi i z_j)$. Define G: $Y_k \to Y_k$ by

$$G(z_1, \dots, z_n) = (z_1 + t_0\phi_{\beta_1} \circ C^{-1}(y_z), \dots, z_n + t_0\phi_{\beta_n} \circ C^{-1}(y_z)) + \boldsymbol{Z}^n$$

where y_z is any point of Y having $z = (z_1, \dots, z_n)$ as its component in Y_k. By Lemma 3, since t_0 is chosen so that $nt_0N < 1$, we have that G is one-to-one. Since $y_k \neq y'_k$, $G(y_k) \neq G(y'_k)$, i.e. $\beta_j(y_k + u_{t_0} \circ C^{-1}(y)) \neq \beta_j(y'_k + u_{t_0} \circ C^{-1}(y'))$ for some j. Therefore $y + u_{t_0} \circ C^{-1}(y) \neq y' + u_{t_0} \circ C^{-1}(y')$, i.e. $g_{t_0} \circ C^{-1}(y) \neq g_{t_0} \circ C^{-1}(y')$.

The following is a direct consequence of Theorems 3 and 6.

COROLLARY 6.1. *Let X be a c.c.m.a. group and let Y be a c.c.m.a. n-dimensional group. Let $T = a + A$ be an affine transformation of X and $S = b + B$ an affine transformation of Y for which there exists a continuous mapping h of X onto Y satisfying $hT = Sh$. There exists a nonaffine continuous mapping g of X onto Y such that $gT = Sg$ if and only if there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a root λ, with $|\lambda| = 1$, of the annihilating polynomial of $\hat{Y}$ with respect to B such that $A^p\delta = \delta$ for some $p \geqq 1$ and $\delta(a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p$ for all such p. If h is a homeomorphism, the above conditions are necessary and sufficient for the existence of a nonaffine homeomorphism g of Y such that $gT = Sg$.*

If B is an endomorphism of a c.c.m.a. n-dimensional group Y onto Y then it follows from the ergodicity conditions stated in §1 that B is ergodic if and only if no root of the annihilating polynomial of $\hat{Y}$ with respect to B is a root of unity. Moreover there is an element $\gamma \in \hat{Y}$, $\gamma \neq 1$ such that $B^p\gamma = \gamma$ if and only if the annihilating polynomial of $\hat{Y}$ with respect to B has a pth root of unity as a root.

THEOREM 7. *Let Y be a c.c.m.a. n-dimensional group and let A and B be endomorphisms of Y onto Y. Suppose there exists a continuous mapping h of Y onto Y such that $hA = Bh$. There exists a nonaffine continuous mapping g of Y onto Y such that $gA = Bg$ if and only if A and B are not ergodic. If h is given to be a homeomorphism then there exists a nonaffine homeomorphism g of Y such that $gA = Bg$ if and only if A and B are not ergodic.*

Proof. If there exists a nonaffine continuous mapping g of Y into Y satisfying $gA=Bg$ then Theorem 3 asserts the existence of $\delta \in \hat{Y}$, $\delta \neq 1$, and a root λ of the annihilating polynomial of $\hat{Y}$ with respect to B such that $A^p\delta=\delta$ for some $p \geqq 1$ and $\lambda^p=1$ for all such p. Therefore A and B are not ergodic.

Conversely suppose A and B are not ergodic. Suppose h is affine or there is nothing to prove. Let $h=c+C$, where $c \in Y$ and C is an endomorphism of Y onto Y. If $\hat{Y}$ is considered as an (additive) subgroup of $\boldsymbol{Q}^n$ the nonsingular matrix representing C is a conjugacy between the matrix representing A and the matrix representing B. Hence the annihilating polynomial of $\hat{Y}$ with respect to A is the same as the annihilating polynomial of $\hat{Y}$ with respect to B. Let $\delta \in \hat{Y}$, $\delta \neq 1$, be such that $A^p\delta=\delta$ for some $p \geqq 1$. Let p be the least positive integer for which $A^p\delta=\delta$. Then the annihilating polynomial of $\hat{Y}$ with respect to B has a root λ which is a pth root of unity. The result now follows from Theorem 6.

We now give an example to show that Theorem 6 is false if the assumption that $\hat{Y}$ is finite-dimensional is replaced by the assumption that $\hat{Y}$ is polynomially annihilated by B, i.e. the converse of Theorem 3 is false.

Let E denote the automorphism of the 4-torus $\boldsymbol{K}^4$ determined by the matrix

$$[E] = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{bmatrix}.$$

The matrix $[E]$ has two eigenvalues λ_1, $\bar{\lambda}_1$ of unit modulus which are not roots of unity and two distinct real eigenvalues λ_2, λ_3 [10]. Let W denote the one-sided direct sum of an infinite number of copies of $\boldsymbol{K}^4$. Let $Y=\boldsymbol{K}+W$. Y is an infinite-dimensional torus. Let $S=b+B$: $Y \to Y$ be defined by

$$S(y_0, y_1, y_2, \ldots) = (b_0, 0, 0, \ldots)+(y_0, Ey_1, y_1+Ey_2, y_2+Ey_3, \ldots)$$

$y_0 \in \boldsymbol{K}$, $y_i \in \boldsymbol{K}^4$ $(i \geqq 1)$, where $\exp[2\pi i b_0]=\lambda_1$. It is not difficult to show that S is ergodic and $\hat{Y}$ is polynomially annihilated by B. The characteristic polynomial of $[E]$ is the minimal annihilating polynomial with respect to B of some of the elements of $\hat{Y}$ and λ_1 is a root of this polynomial. If $\delta \in \hat{Y}$ is defined by

$$\delta(y_0, y_1, y_2, \ldots) = \exp[2\pi i y_0],$$

then $B\delta=\delta$ and $\delta(b)=\lambda_1$. Hence (with $X=Y$ and $T=S$) all the assumptions of Theorem 6 (except that Y be finite-dimensional) are satisfied by this example. However, we shall show that every continuous mapping commuting with S is affine.

Suppose $gS=Sg$ where g is continuous. Let g_n $(n \geqq 0)$ be the projection of g onto the nth factor in the representation $Y=\boldsymbol{K}+\boldsymbol{K}^4+\boldsymbol{K}^4+\cdots$. g_0 is a continuous mapping of Y into $\boldsymbol{K}$ and g_n $(n \geqq 1)$ are continuous mappings of Y into $\boldsymbol{K}^4$. We shall show that each g_n $(n \geqq 0)$ is affine and this implies g is affine. By Theorem 1 there exists a homomorphism $\mu_0\colon Y\to \boldsymbol{K}$ and a continuous mapping $\phi_0\colon Y\to R$ such that $g_0(y)=\mu_0(y)+\phi_0(y)+\boldsymbol{Z}$. Since $g_0(Sy)=b_0+g_0(y)$ we have

$$\phi_0(Sy) = \phi_0(y)+a, \qquad y\in Y, \quad \text{where } a\in \boldsymbol{R}.$$

Therefore $\phi_0(y)-\int_Y \phi_0(y)\,dm$ (m denotes Haar measure on Y) is an invariant function under S and therefore constant. Hence ϕ_0 is constant and g_0 is affine. Suppose that some g_n ($n\geqq 1$) is nonaffine. Let k be the least integer for which g_k is nonaffine. By Theorem 1 there exist homomorphisms $\mu_i\colon Y\to \boldsymbol{K}$ and continuous mappings $\phi_i\colon Y\to \boldsymbol{R}$ ($1\leqq i\leqq 4$) such that

$$g_k(y) = \begin{bmatrix}\mu_1(y)+\phi_1(y)\\ \mu_2(y)+\phi_2(y)\\ \mu_3(y)+\phi_3(y)\\ \mu_4(y)+\phi_4(y)\end{bmatrix}+\boldsymbol{Z}^4.$$

Since $gS=Sg$ we have $g_kS=g_{k-1}+Eg_k$ and $g_{k+1}S=g_k+Eg_{k+1}$. Since g_{k-1} is affine the uniqueness in Theorem 1 gives

$$\begin{bmatrix}\phi_1(Sy)\\ \phi_2(Sy)\\ \phi_3(Sy)\\ \phi_4(Sy)\end{bmatrix} = E\begin{bmatrix}\phi_1(y)\\ \phi_2(y)\\ \phi_3(y)\\ \phi_4(y)\end{bmatrix}+e, \qquad y\in Y, \quad \text{where } e\in \boldsymbol{R}^4.$$

Let D, with matrix

$$[D] = \begin{bmatrix}\lambda_1 & 0 & 0 & 0\\ 0 & \bar{\lambda}_1 & 0 & 0\\ 0 & 0 & \lambda_2 & 0\\ 0 & 0 & 0 & \lambda_3\end{bmatrix}$$

be the Jordan normal form of E and let $U\colon \boldsymbol{C}^4\to \boldsymbol{C}^4$ be the linear transformation such that $U^{-1}EU=D$. By the type of argument used in the proof of Theorem 3 it follows that

$$\begin{pmatrix}\phi_1(y)\\ \phi_2(y)\\ \phi_3(y)\\ \phi_4(y)\end{pmatrix} = U\begin{pmatrix}c\delta(y)\\ d\delta^{-1}(y)\\ 0\\ 0\end{pmatrix}+e', \qquad y\in Y,$$

where $c, d\in \boldsymbol{C}$ and $e'\in \boldsymbol{R}^4$. By Theorem 1 again

$$g_{k+1}(y) = \begin{pmatrix}\mu_5(y)+\phi_5(y)\\ \mu_6(y)+\phi_6(y)\\ \mu_7(y)+\phi_7(y)\\ \mu_8(y)+\phi_8(y)\end{pmatrix}, \qquad y\in Y,$$

where $\mu_i: Y \to \boldsymbol{K}$ are homomorphisms and $\phi_i: Y \to \boldsymbol{R}$ are continuous $(5 \leqq i \leqq 8)$. Since $g_{k+1}(Sy) = g_k(y) + Eg_{k+1}(y)$ we have

$$\begin{pmatrix} \phi_5(Sy) \\ \phi_6(Sy) \\ \phi_7(Sy) \\ \phi_8(Sy) \end{pmatrix} = E \begin{pmatrix} \phi_5(y) \\ \phi_6(y) \\ \phi_7(y) \\ \phi_8(y) \end{pmatrix} + U \begin{pmatrix} c\delta(y) \\ d\delta^{-1}(y) \\ 0 \\ 0 \end{pmatrix} + e'', \qquad y \in Y,$$

where $e'' \in \boldsymbol{R}^4$. Apply U^{-1} to this equation and set

$$\begin{pmatrix} f_1(y) \\ f_2(y) \\ f_3(y) \\ f_4(y) \end{pmatrix} = U^{-1} \begin{pmatrix} \phi_5(y) \\ \phi_6(y) \\ \phi_7(y) \\ \phi_8(y) \end{pmatrix}.$$

Then $f_1(Sy) = \lambda_1 f_1(y) + c\delta(y) + c^1$, where $c' \in \boldsymbol{C}$. Since $f_1 \in L^2(Y)$ let $f_1(y) = \sum_i a_i \gamma_i(y)$ (L^2 convergence) where $\gamma_i \in \hat{Y}$ and $\sum_i |a_i|^2 < \infty$. If $\delta = \gamma_{i_0}$ then $a_{i_0}\delta(b) = \lambda_1 a_{i_0} + c$, and since $\delta(b) = \lambda_1$ this gives $c = 0$. Consideration of the equation for f_2 implies $d = 0$. Therefore ϕ_i, $1 \leqq i \leqq 4$, are constant and g_k is affine, a contradiction. Therefore each g_n $(n \geqq 0)$ is affine.

Thus we have shown that every continuous mapping commuting with S is affine.

We shall now state, without proof, a generalization of Theorem 6. If B is an endomorphism of a c.c.m.a. group Y onto Y we denote by $\hat{Y}(B, \lambda)$ the subgroup of $\hat{Y}$ generated by those elements of $\hat{Y}$ whose minimal annihilating polynomials with respect to B have λ as a root.

THEOREM 8. *Suppose X and Y are c.c.m.a. groups, $T = a + A$ an affine transformation of X, and $S = b + B$ an affine transformation of Y such that $\hat{Y}$ is polynomially annihilated by B. Suppose there exists a continuous mapping h of X onto Y such that $hT = Sh$. Also assume there exists $\delta \in \hat{X}$, $\delta \neq 1$, and a complex number λ with $|\lambda| = 1$, such that $\hat{Y}(B, \lambda)$ is a subgroup of $\hat{Y}$ of finite rank with the properties that $A^p\delta = \delta$ for some $p \geqq 1$ and $\delta(a + A(a) + \cdots + A^{p-1}(a)) = \lambda^p$ for all such p. Then there exists a nonaffine continuous mapping g of X onto Y such that $gT = Sg$. If h is a homeomorphism then g can be chosen to be a homeomorphism.*

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